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General theory of stress and Rate of strain ①

Example 2:- $\tau_{ij} = \alpha(\lambda_i \lambda_j' + \lambda_i' \lambda_j)$

where α is a scalar and λ_i, λ_j are unit vectors.

Solution:- Consider x axis along λ_i and z axis normal to both λ_i

The components of λ_i are $(1,0,0)$ and λ_i' $(\lambda_1', \lambda_2', 0)$

since $\tau_{ij} = \alpha(\lambda_i \lambda_j' + \lambda_i' \lambda_j)$ — ①

α is a scalar and λ_i, λ_i' are unit vectors.

Let P be the value of the Principal stress and (l, m, n) be the dc of the Principal direction.

The Principal stresses are

$$l(\tau_{11} - \tau) + m\tau_{12} + n\tau_{13} = 0$$
$$l\tau_{21} + m(\tau_{22} - \tau) + n\tau_{23} = 0$$
$$l\tau_{31} + m\tau_{32} + n(\tau_{33} - \tau) = 0$$

where $l^2 + m^2 + n^2 = 1$ — ②

From relation ① and ②, we have.

$$l(2\alpha\lambda_1' - P) + m\alpha\lambda_2' = 0$$
$$l\alpha\lambda_2' - mP = 0$$
$$-nP = 0$$
— ③

The characteristic equation becomes

$$\begin{vmatrix} 2\alpha\lambda_1' - P & \alpha\lambda_2' & 0 \\ \alpha\lambda_2' & -P & 0 \\ 0 & 0 & P \end{vmatrix} = 0$$

$$\Rightarrow P[-P(2\alpha\lambda_1' - P) - \alpha\lambda_2'(P\alpha\lambda_2')] = 0$$

$$\Rightarrow P[P(2\alpha\lambda_1' - P) + P\alpha^2\lambda_2'^2] = 0$$

$$\Rightarrow \alpha^2 P\left[\frac{P}{\alpha}\left(2\lambda_1' - \frac{P}{\alpha}\right) + \lambda_2'^2\right] = 0$$

$$\Rightarrow P = 0 \quad \text{and} \quad \frac{P}{\alpha}\left(2\lambda_1' - \frac{P}{\alpha}\right) + \lambda_2'^2 = 0$$

$$2\lambda_1'\frac{P}{\alpha} - \frac{P^2}{\alpha^2} + \lambda_2'^2 = 0$$

$$\Rightarrow \left(\frac{P}{\alpha}\right)^2 - 2\lambda_1'\left(\frac{P}{\alpha}\right) - \lambda_2'^2 = 0$$

since $\lambda_1'^2 + \lambda_2'^2 = 1$

$$\Rightarrow \left(\frac{P}{\alpha}\right)^2 - 2\lambda_1'\left(\frac{P}{\alpha}\right) + \lambda_1'^2 = 1$$

$$\Rightarrow \left(\frac{P}{\alpha} - \lambda_1'\right)^2 = 1$$

$$\Rightarrow \left(\frac{P - \alpha\lambda_1'}{\alpha}\right)^2 = 1$$

$$\Rightarrow P - \alpha\lambda_1' = \alpha$$

$$P = \alpha(\lambda_1 \pm 1)$$

There fore the Principal stresses are

$$0, \ \alpha(\lambda_1'+1), \ \alpha(\lambda_1'-1)$$

consider (l_1, m_1, n_1) be the direction cosines of the Principal stress direction correspond to $P=0$

Substituting $P=0$ in relation ③

$$2\lambda_1'\alpha l_1 + m_1\alpha\lambda_2' = 0$$

$$\lambda_2'\alpha l_1 - 0 = 0$$

$$l_1 = 0, \ m_1 = 0$$

$$\frac{m_2}{\lambda_2'} = \frac{1}{\sqrt{2(1+\lambda_1')}}$$

$$\Rightarrow m_2 = \frac{\lambda_2'}{2(1+\lambda_1')} = \sqrt{\left(\frac{1-\lambda_1'}{2}\right)}$$

$$l_2 = \sqrt{\left(\frac{1+\lambda_1'}{2}\right)}, \quad m_2 = \sqrt{\left(\frac{1-\lambda_1'}{2}\right)}, \quad n_2 = 0$$

Consider $(l_3, m_3 n_3)$ be the direction cosines of the Principal stress direction correspond to $P = \alpha(\lambda_1' - 1)$

From Relation (3)

$$\alpha(\lambda_1'+1)l_3 - \alpha\lambda_2' m_3 = 0$$
$$\alpha\lambda_2' l_3 - \alpha(\lambda_1'-1)m_3 = 0$$

with $l_3^2 + m_3^2 + n_3^2 = 1$

$$\Rightarrow \frac{l_3}{\lambda_2'} = \frac{m_3}{\lambda_1'+1} = \frac{n_3}{0} = \frac{1}{\sqrt{2(1+\lambda_1')}}$$

$$\Rightarrow l_3 = \sqrt{\left(\frac{1-\lambda_1'}{2}\right)}$$

$$m_3 = \sqrt{\left(\frac{1+\lambda_1'}{2}\right)}$$

$$n_3 = 0$$

$= \times = \times =$

where $l_1^2+m_1^2+n_1^2=1$

$$n_1^2=1 \Rightarrow n_1=1$$

$$l_1=0,\ m_1=0,\ n_1=1$$

Consider $(l_2\, m_2\, n_2)$ be the direction cosine of the Principal stress direction corresponding to $P=\alpha(\lambda_1'+1)$

From relation ③

$$l_2(2\lambda_1'\alpha-\lambda_1'\alpha-\cancel{\alpha})+m_2\alpha\lambda_2'=0$$

$$l_2\alpha\lambda_2'-m_2\alpha(\lambda_1'+1)=0$$

$$-n_2\alpha(\lambda_1'+1)=0$$

$$\Rightarrow \quad l_2\alpha(\lambda_1'-1)+m_2\alpha\lambda_2'=0$$

$$l_2\alpha\lambda_2'-m_2(\lambda_1'+1)\alpha=0$$

$$-n_2(\lambda_1'+1)\alpha=0$$

with $l_2^2+m_2^2+n_2^2=1$

$$\frac{l_2}{\lambda_1'+1}=\frac{m_2}{\lambda_2'}=\frac{n_2}{0}$$

$$=\frac{1}{\sqrt{\lambda_2'^2+(\lambda_1'+1)^2}}=\frac{1}{\sqrt{\lambda_2'^2+\lambda_1'^2+1+2\lambda_1'}}$$

$$=\frac{1}{\sqrt{(2+2\lambda_1')}}=\frac{1}{\sqrt{2(1+\lambda_1')}}$$

$$\frac{l_2}{\lambda_1'+1}=\frac{1}{\sqrt{2(1+\lambda_1')}}$$

$$\Rightarrow l_2=\frac{1+\lambda_1'}{\sqrt{2(1+\lambda_1')}}=\sqrt{\left(\frac{1+\lambda_1'}{2}\right)}$$